On the time periodic solutions and the asymptotic stabilities of GFDs and the related equations

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Geophysical Fluid Dynamics

1. Oceanic fluid is made up of a slightly compressible fluid with Coriolis force.
2. Often described by Navier-Stokes equation, Boussinesq equation and primitive equation.
3. Important characteristics : stratification, rotation, temporal periodicity, steady states
4. Only the stable flows could be observed in real world or numerical experiments.

Time periodic flows

(A) (Bona-Hsia-Ma-Wang, 2011) The Hopf bifurcation diagrams of the double-diffusive equation (Boussinesq equation coupled with temperature diffusion and salinity diffusion) is clearly classified according to the regions in the phase space of temperature Rayleigh number and salinity Rayleigh number under suitable physical conditions. This demonstrates a mechanism that produces time periodic circulations due to stratification.

(B) (Hsia-Shiue, 2013; Hsia, Jung, Kwon, Nguyen, Chen, Shiue) The analysis for Navier-Stokes equations and viscous Burgers' equations demonstrates the existence of time periodic flows due to the time periodic external force. The rigorous mathematical analysis shows that there exists at least one (stable or unstable) time periodic flow with the presence of time periodic force in the GFD system.

The effects of time periodic force in GFD

Let f denote the amplitude of the external force in certain appropriate norm. There exists two positive numbers $0 < f_1 < f_2$ such that

1. For $0 < f < f_1$, the time periodic flow is temporal asymptotically stable (Hence, the time periodic flow is unique.).
2. In case $f_1 < f < f_2$, the numerical experiments show that there exist several locally temporal asymptotically stable time periodic flows.
3. If $f > f_2$, the numerical experiments show that there does not exist stable flows.

Physical Implications

Intuitively speaking, it is reasonable to expect that the time periodic external force produces the time periodic flows. This is verified by rigorous mathematics in our analysis for a wide class of model equations. However, while the force is too large ($f > f_2$), the time periodic flows lose its stabilities which means it cannot be captured by physical or numerical experiments. Only the flows generated by small time periodic force ($f < f_2$)could be observed.

Large-scale moist atmosphere model (primitive equation)

$$\frac{\partial \boldsymbol{v}}{\partial t} + \nabla_{\boldsymbol{v}} \boldsymbol{v} + w \frac{\partial \boldsymbol{v}}{\partial \xi} + \frac{f}{R_0} \mathbf{k} \times \boldsymbol{v} + \nabla \Phi = \nu_1 \triangle \boldsymbol{v} + \mu_1 \frac{\partial^2 \boldsymbol{v}}{\partial \xi^2} + F_1, \tag{1}$$

$$\frac{\partial \Phi}{\partial \xi} = -\frac{bP}{p}(1 + aq)T, \tag{2}$$

$$div\, \boldsymbol{v} + \frac{\partial w}{\partial \xi} = 0, \tag{3}$$

$$\frac{\partial T}{\partial t} + \nabla_{\boldsymbol{v}} T + w \frac{\partial T}{\partial \xi} - \frac{bP}{p}(1 + aq)w = \nu_2 \triangle T + \mu_2 \frac{\partial^2 T}{\partial \xi^2} + F_2, \tag{4}$$

$$\frac{\partial q}{\partial t} + \nabla_{\boldsymbol{v}} q + w \frac{\partial q}{\partial \xi} = \nu_3 \triangle q + \mu_3 \frac{\partial^2 q}{\partial \xi^2} + F_3, \tag{5}$$

The Primitive Equation

1. $\boldsymbol{v} = (v_\theta, v_\varphi)$ is the horizontal velocity
2. w is the vertical velocity in p-coordinate system
3. Φ is the geopotential
4. T is the temperature and q is the mixing ratio of water vapor in the air
5. P is an approximate value of the pressure at the surface of the Earth
6. p_0 is the pressure of the upper atmosphere $(p_0 > 0)$
7. the variable ξ satisfies $p = (P - p_0)\xi + p_0\,(p_0 \le p \le P)$.

$$\mathcal{M} = S^2 \times (0,1),$$

The boundary conditions supplemented to the system (1)-(5) are

$$\text{on } \xi = 1\,(p = P):\ \frac{\partial \boldsymbol{v}}{\partial \xi} = 0,\ w = 0,\ \frac{\partial T}{\partial \xi} = -\alpha T,\ \frac{\partial q}{\partial \xi} = -\beta q, \quad (6)$$

$$\text{on } \xi = 0\,(p = p_0):\ \frac{\partial \boldsymbol{v}}{\partial \xi} = 0,\ w = 0,\ \frac{\partial T}{\partial \xi} = 0,\ \frac{\partial q}{\partial \xi} = 0. \quad (7)$$

Here, α and β are given positive constants.

$p-$coordinate

Let e_θ, e_φ, e_ξ be the unit vectors in θ-, φ- and ξ- directions of the space domain $\mathcal{M}$ respectively which are defined by

$$e_\theta = \frac{\partial}{\partial \theta}, \ e_\varphi = \frac{1}{\sin\theta}\frac{\partial}{\partial \varphi}, \ e_\xi = \frac{\partial}{\partial \xi}.$$

Operators

$$div\boldsymbol{v} = div(\boldsymbol{v}_\theta e_\theta + \boldsymbol{v}_\varphi e_\varphi) = \frac{1}{\sin\theta}\left(\frac{\partial}{\partial\theta}(\sin\theta\boldsymbol{v}_\theta) + \frac{\partial\boldsymbol{v}_\varphi}{\partial\varphi}\right), \tag{8}$$

$$\nabla g = \frac{\partial g}{\partial\theta}e_\theta + \frac{1}{\sin\theta}\frac{\partial g}{\partial\varphi}e_\varphi, \tag{9}$$

$$\begin{aligned}\nabla_{\boldsymbol{v}}\tilde{\boldsymbol{v}} &= \left(\boldsymbol{v}_\theta\frac{\partial\tilde{\boldsymbol{v}}_\theta}{\partial\theta} + \frac{\boldsymbol{v}_\varphi}{\sin\theta}\frac{\partial\tilde{\boldsymbol{v}}_\theta}{\partial\varphi} - \boldsymbol{v}_\varphi\tilde{\boldsymbol{v}}_\varphi\cot\theta\right)e_\theta \\ &+ \left(\boldsymbol{v}_\theta\frac{\partial\tilde{\boldsymbol{v}}_\varphi}{\partial\theta} + \frac{\boldsymbol{v}_\varphi}{\sin\theta}\frac{\partial\tilde{\boldsymbol{v}}_\varphi}{\partial\varphi} + \boldsymbol{v}_\varphi\tilde{\boldsymbol{v}}_\varphi\cot\theta\right)e_\varphi,\end{aligned} \tag{10}$$

$$\nabla_{\boldsymbol{v}} g = \boldsymbol{v}_\theta \frac{\partial g}{\partial \theta} + \frac{\boldsymbol{v}_\varphi}{\sin\theta}\frac{\partial g}{\partial \varphi}, \tag{11}$$

$$\triangle g = div(\nabla g) = \frac{1}{\sin\theta}\Big[\frac{\partial}{\partial\theta}\Big(\sin\theta\frac{\partial g}{\partial\theta}\Big) + \frac{1}{\sin\theta}\frac{\partial^2 g}{\partial\varphi^2}\Big], \tag{12}$$

$$\triangle \boldsymbol{v} = \Big(\triangle \boldsymbol{v}_\theta - \frac{2\cos\theta}{\sin^2\theta}\frac{\partial \boldsymbol{v}_\varphi}{\partial\varphi} - \frac{\boldsymbol{v}_\theta}{\sin^2\theta}\Big)e_\theta + \Big(\triangle \boldsymbol{v}_\varphi + \frac{2\cos\theta}{\sin^2\theta}\frac{\partial \boldsymbol{v}_\theta}{\partial\varphi} - \frac{\boldsymbol{v}_\varphi}{\sin^2\theta}\Big)e_\varphi, \tag{13}$$

The boundary condition (7) guarantees

$$\int_0^1 div\boldsymbol{v}\, d\xi = 0. \tag{14}$$

$$w(\theta, \varphi, \xi, t) = W(\boldsymbol{v}) = \int_\xi^1 div\boldsymbol{v}(\theta, \varphi, \xi', t)\, d\xi', \tag{15}$$

$$\Phi(\theta, \varphi, \xi, t) = \Phi_1(\theta, \varphi, t) + \int_\xi^1 \frac{bP}{p}(1 + aq)T(\theta, \varphi, \xi', t)\, d\xi'. \tag{16}$$

Then, we obtain the following system for $(\boldsymbol{v}, T, q)$:

$$\frac{\partial \boldsymbol{v}}{\partial t} + \nabla_{\boldsymbol{v}} \boldsymbol{v} + W(\boldsymbol{v}) \frac{\partial \boldsymbol{v}}{\partial \xi} + \frac{f}{R_0} \mathbf{k} \times \boldsymbol{v} + \nabla \Phi_1 + \int_{\xi}^{1} \frac{bP}{p} \nabla [(1 + aq)T] \, d\xi'$$

$$= \nu_1 \triangle \boldsymbol{v} + \mu_1 \frac{\partial^2 \boldsymbol{v}}{\partial \xi^2} + F_1, \tag{17}$$

$$\frac{\partial T}{\partial t} + \nabla_{\boldsymbol{v}} T + W(\boldsymbol{v}) \frac{\partial T}{\partial \xi} - \frac{bP}{p} (1 + aq) W(\boldsymbol{v}) = \nu_2 \triangle T + \mu_2 \frac{\partial^2 T}{\partial \xi^2} + F_2, \tag{18}$$

$$\frac{\partial q}{\partial t} + \nabla_{\boldsymbol{v}} q + W(\boldsymbol{v}) \frac{\partial q}{\partial \xi} = \nu_3 \triangle q + \mu_3 \frac{\partial^2 q}{\partial \xi^2} + F_3, \tag{19}$$

$$\int_0^1 div \boldsymbol{v} \, d\xi = 0. \tag{20}$$

Boundary Condition

$$\text{on } \xi = 1 : \frac{\partial \boldsymbol{v}}{\partial \xi} = 0,\ \frac{\partial T}{\partial \xi} = -\alpha T,\ \frac{\partial q}{\partial \xi} = -\beta q, \tag{21}$$

$$\text{on } \xi = 0 : \frac{\partial \boldsymbol{v}}{\partial \xi} = 0,\ \frac{\partial T}{\partial \xi} = 0,\ \frac{\partial q}{\partial \xi} = 0. \tag{22}$$

Averaged equation

$$\frac{\partial \bar{\boldsymbol{v}}}{\partial t} + \nabla_{\bar{\boldsymbol{v}}} \bar{\boldsymbol{v}} + \overline{\tilde{\boldsymbol{v}} div \tilde{\boldsymbol{v}} + \nabla_{\tilde{\boldsymbol{v}}} \tilde{\boldsymbol{v}}} + \frac{f}{R_0} \mathbf{k} \times \bar{\boldsymbol{v}} + \nabla \Phi_1$$

$$+ \overline{\int_{\xi}^{1} \frac{bP}{p} \nabla [(1 + aq)T] \, d\xi'} = \nu_1 \triangle \bar{\boldsymbol{v}} + \bar{F}_1, \tag{23}$$

$$div \bar{\boldsymbol{v}} = 0, \tag{24}$$

Fluctuation

$$\frac{\partial \tilde{\boldsymbol{v}}}{\partial t} + \nabla_{\tilde{\boldsymbol{v}}}\tilde{\boldsymbol{v}} + W(\tilde{\boldsymbol{v}})\frac{\partial \tilde{\boldsymbol{v}}}{\partial \xi} + \nabla_{\tilde{\boldsymbol{v}}}\bar{\boldsymbol{v}} + \nabla_{\bar{\boldsymbol{v}}}\tilde{\boldsymbol{v}} - \overline{(\tilde{\boldsymbol{v}}\, div\tilde{\boldsymbol{v}} + \nabla_{\tilde{\boldsymbol{v}}}\tilde{\boldsymbol{v}})} + \frac{f}{R_0}\mathbf{k}\times\tilde{\boldsymbol{v}}$$

$$+ \int_\xi^1 \frac{bP}{p}\nabla[(1+aq)T]\, d\xi' - \overline{\int_\xi^1 \frac{bP}{p}\nabla[(1+aq)T]\, d\xi'}$$

$$= \nu_1 \triangle \tilde{\boldsymbol{v}} + \mu_1 \frac{\partial^2 \tilde{\boldsymbol{v}}}{\partial \xi^2} + \tilde{F}_1, \tag{25}$$

$$\text{on } \xi = 1: \frac{\partial \tilde{\boldsymbol{v}}}{\partial \xi} = 0, \tag{26}$$

$$\text{on } \xi = 0: \frac{\partial \tilde{\boldsymbol{v}}}{\partial \xi} = 0. \tag{27}$$

Function Space

$$H = H_1 \times H_2 \times H_3, \quad V = V_1 \times V_2 \times V_3,$$

$$\tilde{V}_1 = \Big\{ \boldsymbol{v} \in C^\infty(T\mathcal{M}|TS^2)^2 : \int_0^1 div \boldsymbol{v} \, d\xi = 0,$$

$$\frac{\partial \boldsymbol{v}}{\partial \xi}\Big|_{\xi=0} = \frac{\partial \boldsymbol{v}}{\partial \xi}\Big|_{\xi=1} = 0 \Big\},$$

$$\tilde{V}_2 = \left\{ T \in C^\infty(T\mathcal{M}|TS^2) : \frac{\partial T}{\partial \xi}\Big|_{\xi=0} = 0, \ \frac{\partial T}{\partial \xi}\Big|_{\xi=1} = -\alpha T \right\},$$

$$\tilde{V}_3 = \left\{ q \in C^\infty(T\mathcal{M}|TS^2) : \frac{\partial q}{\partial \xi}\Big|_{\xi=0} = 0, \ \frac{\partial q}{\partial \xi}\Big|_{\xi=1} = -\beta q \right\},$$

Function Space

$$
\begin{aligned}
V_1 &= \text{ the closure of } \tilde{V}_1 \text{ in the norm } ||\cdot||_1, \\
V_2 &= \text{ the closure of } \tilde{V}_2 \text{ in the norm } ||\cdot||_1, \\
V_3 &= \text{ the closure of } \tilde{V}_3 \text{ in the norm } ||\cdot||_1, \\
H_1 &= \text{ the closure of } \tilde{V}_1 \text{ in the norm } |\cdot|_2, \\
H_2 &= H_3 = L^2(\mathcal{M}).
\end{aligned}
$$

Theorem (Guo and Huang)

Let $F_1 = 0$, $(F_2, F_3) \in H^1(\mathcal{M})^2$, $(\boldsymbol{v}_0, T_0, q_0) \in V_1 \times V_2 \times V_3$ and $\mathcal{T} > 0$, then there exists a unique strong solution $(\boldsymbol{v}, T, q)$ to the system of 3D viscous primitive equations of the large-scale moist atmosphere on the interval $[0, \mathcal{T}]$, which depends on the initial data continuously in $H_1 \times H_2 \times H_3$.

Theorem

Assume $F = (F_1, F_2, F_3) \in L^\infty(0, \infty; L^2(\mathcal{M})^4)$, $\partial F/\partial \xi \in L^\infty(0, \infty; L^2(\mathcal{M})^4)$, $(\boldsymbol{v}_0, T_0, q_0) \in V$, $\partial \boldsymbol{v}_0/\partial \xi \in L^4(\mathcal{M})^2$, $\partial T_0/\partial \xi$ and $\partial q_0/\partial \xi \in L^4(\mathcal{M})$. For $(\boldsymbol{v}, T, q)$, the strong solution to (17)-(22) with IC $(\boldsymbol{v}_0, T_0, q_0)$ such that if $(\boldsymbol{v}_0, T_0, q_0)$ and $F = (F_1, F_2, F_3)$ satisfies

$$
\begin{aligned}
& ||\boldsymbol{v}_0||_1^2 + ||T_0||_1^2 + ||q_0||_1^2 + |\frac{\partial \boldsymbol{v}_0}{\partial \xi}|_4^2 + |\frac{\partial T_0}{\partial \xi}|_4^2 + |\frac{\partial q_0}{\partial \xi}|_4^2 \\
& \qquad \le \gamma_1 \le \gamma_1^*, \qquad (28) \\
& |F|^2_{L^\infty(0,\infty;L^2(\mathcal{M})^4)} + |\frac{\partial F}{\partial \xi}|^2_{L^\infty(0,\infty;L^2(\mathcal{M})^4)} \le \gamma_2 \le \gamma_2^*, \qquad (29)
\end{aligned}
$$

Then we have

$$\sup_{t\geq 0}\left\{||\boldsymbol{v}(t)||_1^2+||T(t)||_1^2+||q(t)||_1^2+|\frac{\partial \boldsymbol{v}(t)}{\partial \xi}|_4^2+|\frac{\partial T(t)}{\partial \xi}|_4^2\right.$$
$$\left.+|\frac{\partial q(t)}{\partial \xi}|_4^2\right\}\leq K(\gamma_1,\gamma_2),$$

Theorem

Suppose $F=(F_1,F_2,F_3), \partial F/\partial\xi \in L^\infty(0,\infty;L^2(\mathcal{M})^4)$. There exists a positive number $\tilde{\gamma}_2$ such that if

$$|F|^2_{L^\infty(0,\infty;L^2(\mathcal{M})^4)} + |\frac{\partial F}{\partial \xi}|^2_{L^\infty(0,\infty;L^2(\mathcal{M})^4)} \le \tilde{\gamma}_2, \tag{30}$$

then for any two strong solutions $(\boldsymbol{v}_1(t), T_1(t), q_1(t))$ and $(\boldsymbol{v}_2(t), T_2(t), q_2(t))$ of the system (14)-(20), we have

$$\lim_{t\to\infty}\left(|\boldsymbol{v}_1(t)-\boldsymbol{v}_2(t)|_2^2+|T_1(t)-T_2(t)|_2^2+|q_1(t)-q_2(t)|_2^2\right)=0. \tag{31}$$

The convergence rate in (31) is exponential.

Theorem

Let $F \in L^\infty(0,\infty; L^2(\mathcal{M})^4) \cap C(0,\infty; L^2(\mathcal{M})^4)$ and $\partial F/\partial\xi \in L^\infty(0,\infty; L^2(\mathcal{M})^4)$ be nontrivial and periodic in time with period $\mathcal{T}$. There exists a constant $\tilde{\gamma}_2$ depending on the diffusivity coefficients ν_i, μ_i, $i = 1, 2, 3$, the boundary condition α, β and the size of domain such that if

$$|F|^2_{L^\infty(0,\infty;L^2(\mathcal{M})^4)} + |\frac{\partial F}{\partial \xi}|^2_{L^\infty(0,\infty;L^2(\mathcal{M})^4)} \le \tilde{\gamma}_2, \tag{32}$$

then there exists a time periodic strong solution $(\boldsymbol{v}, T, q)$ to the system (17)-(22). Moreover, any other strong solution tends to this time-periodic solution asymptotically in L^2 sense.

Serrin 1959

Reynolds number $= Vd/\nu$. Suppose the force term is a time periodic function and the Navier-Stokes equation has a solution with Reynolds number less than 5.71 and this solution is equi$-$continuous in space variables for all the time $t > 0$. Then there exists a asymptotic stable time periodic solution to the Navier-Stokes equation.

The Burgers' type Equation

$$u_t - \nu u_{xx} + u u_x = F, \ x \in (0,1), \ t \geq 0, \tag{33}$$

$$F(x,t) = F(x, t+T).$$

$$u(0,t) = u(1,t) = 0, \ t \geq 0, \tag{34}$$

Look for solutions that satisfy

$$u(x,t) = u(x, t+T), \ x \in [0,1]. \tag{35}$$

Abstract form

$$\frac{du}{dt} + \nu Au + B[u,u] = F,\ \ t \geq 0, \tag{36}$$

$$u(0) = u(T). \tag{37}$$

$$F(t) \in C^1(0,T;H^1(0,1)^2) \text{ and } F(0) = F(T), \tag{38}$$

and let

$$M_* = \sup_{0\leq t\leq T} |F(t)|^2_{L^2} + \sup_{0\leq t\leq T} |F'(t)|^2_{L^2}. \tag{39}$$

Galerkin Method

$$Ae_j = \lambda_j e_j, \; j = 1, 2, \cdots .$$

$$e_j \in H_0^1(0,1).$$

$$u_m(x,t) = \sum_{j=1}^{m} d_j(t) e_j(x).$$

$$\mathcal{V}_m = span\{e_1, e_2, \cdots, e_m\}$$

Theorem

Suppose $F \in C([0,T]; H^1(0,1))$*, and* M_* *is given in* (39)*. Then there exists a time periodic strong solution* u *belonging to*

$$C^1([0,T] \times [0,1]) \cap L^\infty(0,T; D(A))$$

of (33)*. Moreover, there exists* $M_0 > 0$ *such that if* $M_* \in (0, M_0)$*, then the time periodic solution with period* T *of* (33) *is unique.*

Theorem

For each m*, there exists a solution* $u_m \in C^1([0,T];\mathcal{V}_m)$ *of the nonlinear differential equation*

$$\begin{cases} \left\langle \dfrac{du_m}{dt}, e_j \right\rangle + \nu a(u_m, e_j) + b(u_m, u_m, e_j) = \langle F, e_j \rangle, \\ u_m(0) = u_m(T), \end{cases} \tag{40}$$

Lemma

For fixed $w \in C([0,T];\mathcal{V}_m)$, there exists a unique solution $\tilde{u}_m \in C^1([0,T];\mathcal{V}_m)$ of the equation

$$\begin{cases} \left\langle \frac{d\tilde{u}_m}{dt}, \zeta \right\rangle + \nu a(\tilde{u}_m, \zeta) + b(w, w, \zeta) = \langle F, \zeta \rangle, \ \forall \zeta \in \mathcal{V}_m \\ \tilde{u}_m(0) = \tilde{u}_m(T). \end{cases} \tag{41}$$

p.f.

$$\begin{cases} \frac{d}{dt}\tilde{\mathbf{u}}(t) + \tilde{\mathbf{A}}\tilde{\mathbf{u}}(t) = \tilde{\mathbf{F}}(t), \\ \tilde{\mathbf{u}}(0) = \tilde{\mathbf{u}}(T), \end{cases}$$

$$\tilde{\mathbf{u}}(t) = e^{-t\tilde{\mathbf{A}}}\tilde{\mathbf{u}}(0) + \int_0^t e^{(\tau-t)\tilde{\mathbf{A}}}\tilde{\mathbf{F}}(\tau)\, d\tau.$$

$$\tilde{\mathbf{u}}(0) = \tilde{\mathbf{u}}(T) = e^{-T\tilde{\mathbf{A}}}\tilde{\mathbf{u}}(0) + \int_0^T e^{(\tau-T)\tilde{\mathbf{A}}}\tilde{\mathbf{F}}(\tau)\, d\tau$$

Theorem (Schaefer's Fixed Point Theorem)

Let X be a Banach space. $\Phi : X \to X$ is a continuous and compact mapping. Assume that the set

$$\{u \in X \big| u = \lambda\Phi(u) \textit{ for some } 0 \leq \lambda \leq 1\}$$

is bounded. Then Φ has a fixed point.

A Priori Estimates

There exists a constant $C(T, M_*, \nu)$ independent of n such that

$$\sup_{t\in\mathbb{R}} |u_n(t)|_{L^2}^2 \le C(T, M_*, \nu), \tag{42}$$

$$\sup_{t\in\mathbb{R}} \|u_n(t)\|^2 \le C(T, M_*, \nu), \tag{43}$$

$$\sup_{t\in\mathbb{R}} |u_n'(t)|_{L^2}^2 \le C(T, M_*, \nu), \tag{44}$$

$$\int_0^T |Au_n|_{L^2}^2 dt \le C(T, M_*, \nu), \tag{45}$$

$$\int_0^T \|u_n'\|^2 dt \le C(T, M_*, \nu), \tag{46}$$

for each solution u_n of (40), $n = 1, 2, 3 \cdots$.

A Priori Estimates

$$\sup_{t\in\mathbb{R}} |Au_n|_{L^2}^2 \le C(T, M_*, \nu), \tag{47}$$

$$\sup \|u_n'(t)\|^2 \le C(T, M_*, \nu), \tag{48}$$

$$\int_0^T |u_n''(t)|_{L^2}^2 dt \le C(T, M_*, \nu), \tag{49}$$

for each solution u_n of (40), $n = 1, 2, 3 \cdots$.

Theorem

There exists a constant M_0 such that if $M_* \in (0, M_0)$, the time periodic solution with period T of (33) is globally asymptotically stable in V. Namely, any solution of (33) tends to this time periodic solution in H^1 as $t \to \infty$ exponentially.

Thank you very much for your attention!!